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Generalización fraccional de la ecuación de Schrodinger relacionada a la Mecánica Cuántica

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Resumen

El objeto de este trabajo es presentar una solución computacional de una generalización fraccional unidimensional de la ecuación de Schrodinger, relacionada a la Mecánica Cuántica. El método utiliza conjuntamente la transformada de Sumudu y la transformada de Fourier. La solución es obtenida en forma computacional y cerrada en términos de la función de Mittag-Leffler y la función H. El resultado principal obtenido aquí es general y, a partir de éste, se pueden deducir un gran número de casos especiales, hasta ahora dispersos en la literatura. Además, éste provee una extensión de un resultado dado anteriormente por Debnath, Saxena y Chaurasia. El resultado principal es presentado en forma de Teorema y se mencionan varios casos especiales.

Palabras clave: Funciòn de Mittag-Leffler, Funciòn-H, Transformada Sumudu, Transformada de Laplace, Derivada de Caputo

Fractional generalization of Schrödinger equation related to Quantum Mechanics

Abstract

The object of this article is to present the computational solution of a linear one-dimensional fractional generalization of Schrödinger equation occurring in quantum mechanics. The method followed is that of joint Sumudu transform and Fourier transform. The solution is derived in a closed and computational form in terms of the Mittag-Leffler function and the H-function. The main result derived here is general in nature and capable of yielding a large number of special cases hitherto scattered in literature. It also provides an extension of a result given earlier by Debnath, Saxena et al. and Chaurasia et al. The main result is presented in the form of a Theorem, and several special cases are mentioned.

Key words: Mittag-Leffler function, H-function, Sumudu transform, Laplace transform, Caputo derivative

Mathematics Subject Classification 2010: 26A33, 44A10, 33C60, 35J10

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1. Introduction

Fractional differential equations are the generalizations of the ordinary differential equations to arbitrary order (real or complex). During last two decades, more interest is developed by various research workers in formulating fractional differential equations, due to their usefulness and capability to model and solve complex systems. In this connection, one can refer to [7, 9, 16, 17, 21, 23, 26 and 29].

Fractional Schrödinger equation is a fundamental equation of quantum mechanics. This equation is discussed by Laskin [18, 19 and 20] in an attempt to investigate a generalization of Feynman path integrals from Brownian –like to Lévy–like quantum mechanical paths. Earlier Feynman and Hibbs [12] reconstructed the Schrödinger equation by making use of the path integral approach and making use of the well –known Gaussian probability distribution.

The Schrödinger equation thus obtained contains space and time fractional derivatives. In a similar manner, one obtains a time fractional equation if non-Marcovian evolution is considered. In a recent paper, Naber [25] discussed certain properties of time fractional Schrödinger equation by expressing the Schrödinger equation in terms of fractional derivatives as dimensionless objects. Time fractional Schrödinger equations are also discussed by Debnath [6], Bhatti [3], and Debnath and Bhatti [8].

In a recent paper, the authors have investigated the solution of the following generalized linear one dimensional fractional Schrödinger equation of a free particle of mass m, defined by

$$\frac{\partial^{\alpha} N}{\partial t^{\alpha}} = (i\hbar/2m)\,\frac{\partial^{\beta}}{\partial x^{\beta}} N(x,t), -\infty < x < \infty, t > o \qquad 0 < \alpha \le 1, \beta > 0 \tag{1}$$

$$N(x,0) = N_0(x), \ -\infty < x < \infty \tag{2}$$

$$N(x,t) \to 0 \ \text{ as } |x| \to \infty, \tag{3}$$

where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative defined by (15) and $\frac{\partial^{\beta}}{\partial x^{\beta}}$ is the Liouville fractional space derivative, defined by (21), N(x, t) is the wave function, $h = 2\pi\hbar = 6.625 \times 10^{-27}$ *erg sec* = 4.14×10^{-21} *MeV sec*., is the Planck constant and $N_0(x)$ is an arbitrary function. The above defined Schrodinger equation is further generalized recently by Saxena et al. [35] by employing the Hilfer fractional derivative [16, p.113, eq. (105)], instead of the Caputo derivative, defined by (15).

Probability structure of time fractional Schrödinger equation is recently discussed by Tofight [36]. Some physical applications of fractional Schrödinger equation are investigated by Guo and Xu [13] by deriving the solution for a free particle and infinite square potential well. This has motivated the authors to investigate the solution of a fractional generalization of Schrödinger equation (26) in one-dimension ,occurring in quantum mechanics.

Fractional reaction- diffusion equations are treated by Haubold et al. [14], Saxena et al. [31, 32, 33] and Henry and Wearne [15].

2. Mathematical prerequisites

Sumudu transform is defined by [1]

$$\text{G(u)} = f^{\sim}(u) = \text{S[f(t);u]} = \int_0^{\infty} f(ut) e^{-t}\, dt, \qquad u \in (-\tau_1, -\tau_2) \quad , \tag{4}$$

over the set of functions

$$A = \{ f(t) \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \ t \in (-1)^j \times [0, \infty) \} \ , \tag{5}$$

where G(u) is called the Sumudu transform of f(t). It is clear that it is a linear operator. It is a slight variant of the well-known Laplace transform. Further Sumudu transform preserves the unit and scale properties, which makes it an ideal tool for solving several problems of physical & engineering sciences without resorting to a new frequency domain. The relations connecting the Sumudu transform and Laplace transform defined by (16) are given by the following Theorem:

Theorem 2.1 [1, p. 105]. Let $f(t) \in A$ with the Laplace transform F(s). Then the Sumudu transform G(u) of f(t) is given by

$$G(u) = \frac{1}{u} F(\frac{1}{u}) . \tag{6}$$

Corollary 2.1 For, the Sumudu transform of $t^{\rho-1}$ is given by

$$G(u) = S(t^{\rho-1};u) = \Gamma(\rho)u^{\rho-1}. \ (\Re(\rho) > 0, \Re(u) > 0) \tag{7}$$

Further, we also have

$$S^{-1}(u^{\rho-1};t) = \frac{t^{\rho-1}}{\Gamma(\rho)} . \ (\Re(\rho) > 0, \Re(u) > 0) \tag{8}$$

Corollary 2.2 Let $f(t) \in A$. Re(s)>0, and F and G are the Laplace transform and the Sumudu transform of the function f respectively, then

$$F(s) = \frac{1}{s} G(\frac{1}{s}) . \tag{9}$$

Lemma 2.1

$$S^{-1}[u^{\gamma-1}(1-\omega u^{\beta})^{-\delta};t] = t^{\gamma-1}E^{\delta}_{\beta,\gamma}(\omega t^{\beta}). \tag{10}$$

where $\Re(\gamma) > 0, \Re(u) > 0, |\omega u^{\beta}| < 1,$ and $E^{\delta}_{\beta,\gamma}(\omega t^{\beta})$ is the generalized Mittag-Leffler function defined by Prabhakar [28] in the form

$$E^{\delta}_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(n\beta+\gamma)(n)!} , \tag{11}$$

with $z, \beta, \gamma, \delta \in C, \min\{\Re(\beta), \Re(\gamma)\} > 0.$

The result (10) can be easily proved by expanding the binomial function and interpreting the result thus obtained by an appeal to the equation (8). It will be seen that this result is directly applicable in the derivation of the solution of the fractional differential equation (26).

Note 2.1 When $\delta = 1,$ (11) reduces to the Mittag-Leffler function studied by Wiman [37] in the following form

$$E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta+\gamma)}, , \tag{12}$$

where $\beta, \gamma, z \in C, \min\{\Re(\beta), \Re(\gamma)\} > 0$

For $\beta, z \in C, \min\{\Re(\beta), \Re(\gamma)\} > 0$ $\gamma = 1$, (12) reduces to the Mittag-Leffler function [11, 24]:

$$E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta+1)}, \tag{13}$$

where $\beta, z \in C, \Re(\beta) > 0.$

Lemma 2.2

$$S^{-1}\left[\frac{u^{\rho-1}}{u^{-\alpha}+\eta u^{-\beta}+\xi u^{-\gamma}+b};t\right]$$

$$=\sum_{r=0}^{\infty}(-1)^r\sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell}t^{\alpha+\rho+(\alpha-\beta)r+(\beta-\gamma)\ell-1}E^{r+1}_{\alpha,\alpha+\rho+(\alpha-\beta)r+(\beta-\gamma)\ell}(-bt^{\alpha}), \tag{14}$$

where $\Re(\rho)>0, \Re(u)>0, \Re(\rho+\alpha)>0, \left|\frac{\xi u^{-\gamma}+\eta u^{-\beta}}{u^{-\alpha}+b}\right|, (b=a\,|\,k\,|^{\sigma}),$

Proof: We have

$$\frac{u^{\rho-1}}{u^{-\alpha}+\eta u^{-\beta}+\xi u^{-\gamma}+b}=\frac{u^{\rho-1}}{(u^{-\alpha}+b)\left(1+\frac{\xi u^{-\gamma}+\eta u^{-\beta}}{u^{-\alpha}+b}\right)}$$

$$=\sum_{r=0}^{\infty})-1)^r\sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell}[u^{\alpha+\rho+r(\alpha-\beta)+\ell(\beta-\gamma)-1}(1+bu^{\alpha})^{-(r+1)}]$$

From above, it follows that

$$S^{-1}\left[\frac{u^{\rho-1}}{u^{-\alpha}+\eta u^{-\beta}+\xi u^{-\gamma}+b};t\right]$$

$$=\sum_{r=0}^{\infty}(-1)^r\sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell}S^{-1}\{u^{\alpha+\rho+r(\alpha-\beta)+\ell(\beta-\gamma)-1}(1+bu^{\alpha})^{-(r+1)};t\}$$

Evaluating S^{-1} on the right of above equation, with the help of Lemma 2.1, we arrive at the desired result (14). The term by term inversion is justified in view of the result [**10**, §22].

The following fractional derivative of order $\alpha>0$ is introduced by Caputo [4] in the form

$${}_0^cD_t^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} \quad m-1<\alpha\le m,\ \mathrm{Re}(\alpha)>0,\ m\in N. \tag{15}$$

$$=\frac{d^m f(t)}{dt^m},\ \text{if}\ \alpha=m. \tag{16}$$

where $\frac{d^m}{dt^m}f$ is the m^{th} derivative of order m of the function f(t) with respect to t.

Lemma 2.3 The Sumudu transform of Caputo derivative defined by (15) is given by

$$S\left[{}_0^cD_t^{\alpha}f(t);u\right]=S\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}\right] \tag{17}$$

By the application of the convolution theorem of the Sumudu transform [2], the right hand side of the equation (17) becomes

$$\frac{u}{\Gamma(m-\alpha)} S[f^m(t); u] S[t^{m-\alpha-1}; u] \ . \tag{18}$$

Applying the Sumudu transform of multiple differentiation, we have

$$S\left[{}_0^c D_t^\alpha f(t); u \right] = u^{m-\alpha} \left[\frac{G(u)}{u^m} - \sum_{r=0}^{m-1} \frac{f^r(0)}{u^{m-r}} \right]$$

$$= \frac{G(u)}{u^\alpha} - \sum_{r=0}^{m-1} \frac{f^r(0)}{u^{\alpha-r}}, \tag{19}$$

where G(u) = S[f(t);u]. (20)

The above formula is useful in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion. In this connection, one can refer to the monographs written by Podlubny [29], Samko et al. [30], Kilbas et al. [17] , Mathai et al. [22] and Diethelm [9].

Note 2.2 If there is no confusion, then the derivative ${}_0^c D_t^\alpha$ for simplicity will be denoted by ${}_0 D_t^\alpha$

The Liouville fractional derivative of order α is defined in [30, Section 24.2] in the form

$$\frac{\partial^\alpha}{\partial x^\alpha} N(x,t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\partial}{\partial x} \right)^m \int_{-\infty}^{x} \frac{N(t,y)}{(x-y)^{\alpha-m+1}} dy \tag{21}$$

where $(x \in R, \alpha > 0, \quad (m = [\alpha] + 1)\ [\alpha]$ is the integer part of α .

Note 2.3 The operator defined by (21) is also denoted by ${}_{-\infty} D_x^\alpha N(x,t)$. Its Fourier transform is given in [23, p.59, A.12]

$$F\{{}_{-\infty} D_x^\alpha f(x,t); k\} = \quad -|k|^\alpha \ \Psi(k,t), \quad (\alpha > 0) \tag{22}$$

where $\Psi(k,t)$ is the Fourier transform of f (x, t).

Note 2.4 Applications of fractional calculus in the solution of physical problems can be found in the works [7, 16, 22, 26 and 29].

The H-function is defined by means of a Mellin- Barnes type integral in the following manner [22, p.2]:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right]$$

$$= H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_\Omega \Theta(\xi) z^{-\xi} d\xi \,, \tag{23}$$

where $i = (-1)^{1/2}$,

$$\Theta(\xi) = \frac{\left[\prod_{j=1}^{m} \Gamma(b_j + B_j \xi) \right] \left[\prod_{j=1}^{n} \Gamma(1 - a_j - A_j \xi) \right]}{\left[\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j \xi) \right] \left[(\prod_{j=n+1}^{p} \Gamma(a_j + A_j \xi) \right]} \,, \tag{24}$$

and an empty product is always interpreted as unity ; m, n, p, q $\in N_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in R_+, a_i, b_j \in R$ or C (i=1,…,p ; j=1,…,q) such hat

$$A_i(b_j + k) \neq B_j(a_i - \ell - 1) \ (k, \ell \in N_0; \ \text{i=1,…,n ; j=1,…,m }), \tag{25}$$

where we employ the usual notations: $N_0 = (0,1,2\ldots)$; R = (-∞,∞) , $R_+ = (0,\infty)$, and C being the complex number field.

3. Unified fractional generalization of Schrödinger equation

In this section, the solution of a linear one-dimensional fractional Schrödinger equation (26) is investigated. The result is presented in the form of the following:

Theorem 3.1 Consider the one-dimensional fractional generalization of the Schrödinger equation of a free particle of mass m, defined by

$$\begin{aligned} &{}_0D_t^{\alpha} N(x,t) + \eta\, {}_0D_t^{\beta} N(x,t) + \xi\, {}_0D_t^{\gamma} N(x,t) = \left(\frac{i\hbar}{2m}\right) {}_{-\infty}D_x^{\eta} N(x,t); \\ &(0 < \alpha \le 1, 0 < \beta \le 1, 0 < \gamma \le 1) \end{aligned} \tag{26}$$

with initial conditions

$$\text{N(x, 0) = f(x)}, \quad x \in \Re\,, \quad \lim_{x \to \pm\infty} N(x,t) = 0, t > 0; \eta, \xi \in R^+, \tag{27}$$

where ${}_0D_t^{\alpha}, {}_0D_t^{\beta}, {}_0D_t^{\gamma}$ are the Caputo fractional derivatives of orders $\alpha > 0, \beta > 0, \gamma > 0$, respectively as defined by (15), ${}_{-\infty}D_x^{\sigma}$ is the Liouville partial fractional derivative of order $\sigma > 0$, defined by (21), N(x, t) is the wave function,

$$\begin{aligned} h = 2\pi\hbar &= 6.625 \text{ x } 10^{-27} \ erg\ sec \\ &= 4.14 \text{ x } 10^{-21} \ MeV\ sec \end{aligned}$$

is the Planck constant and f(x) is a prescribed function. Then under the above conditions, there holds the following formula for the solution of (26):

$$\begin{aligned} N(x,t) = \frac{1}{2\pi} \sum_{r=0}^{\infty} (-1)^r \int_{-\infty}^{\infty} e^{-ikx} \sum_{\ell=0}^{r} \binom{r}{\ell} \xi^{\ell} \eta^{r-\ell} f^*(k) \{ t^{(\alpha-\beta)r+(\beta-\gamma)\ell} E^{r+1}_{\alpha,(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha}) \\ + \eta t^{(\alpha-\beta)(r+1)+\ell(\beta-\gamma)} E^{r+1}_{\alpha,(\alpha-\beta)(r+1)+(\beta-\gamma)\ell+1}(-bt^{\alpha}) \\ + \xi t^{\alpha-\gamma+(\alpha-\beta)r+\ell(\beta-\gamma)} E^{r+1}_{\alpha,\alpha\gamma+(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha}) \} dk \quad (b = a\,|\,k\,|^{\sigma}; a = \frac{ih}{2m}) \end{aligned} \tag{28}$$

provided that the series and integrals in (28) are convergent.

Proof: Applying the Sumudu transform with respect to the time variable t and (14) and using the boundary conditions, we find that

$$\begin{aligned} &u^{-\alpha} N^{\sim}(x,u) - u^{-\alpha} f(x) + \eta u^{-\beta} N^{\sim}(x,u) - \eta u^{-\beta} f(x) + \xi s^{-\gamma} N^{\sim}(x,u) - \xi s^{-\gamma} f(x)\,. \\ &= \left(\frac{ih}{2m}\right) {}_{-\infty}D_x^{\eta} N^{\sim}(x,u) \end{aligned} \tag{29}$$

If we apply the Fourier transform with respect to the space variable x and apply the result (22), it yields

$$u^{-\alpha}N^{\sim *}(k,u)-u^{-\alpha}f^{*}(k)+\eta s^{\beta}N^{\sim *}(k,s)-\eta s^{-\beta}f^{*}(k)+\xi u^{-\gamma}N^{\sim *}(k.u)-\xi u^{-\gamma}f^{*}(k)$$
$$= -a\,|k|^{\sigma}\,N^{\sim *}(k,u) \qquad (a=\frac{i\hbar}{2m}) \tag{30}$$

Solving for $N^{\sim *}(k,u)$, it gives

$$N^{\sim *}(k,s) = \frac{(u^{-\alpha}+\eta u^{-\beta}+\xi u^{-\gamma})f^{*}(k)}{u^{-\alpha}+\eta u^{-\beta}+\xi u^{-\gamma}+b}, \tag{31}$$

where $b=a\,|k|^{\sigma}$ ..

To invert the equation (31), it is convenient to first invert the Sumudu transform and after that the Fourier transform. On taking the inverse Sumudu transform of the above expression with the help of the result (14), it is found that

$$N^{*}(k,t)=f^{*}(k)\sum_{r=0}^{\infty}(-1)^{r}\sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell}[t^{(\alpha-\beta)r+(\beta-\gamma)\ell}E^{r+1}_{\alpha,(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha})$$
$$+\eta t^{(\alpha-\beta)(r+1)+\ell(\beta-\gamma)}E^{r+1}_{\alpha,(\alpha-\beta)(r+1)+(\beta-\gamma)\ell+1}(-bt^{\alpha})$$
$$+\ \xi t^{\alpha-\gamma+(\alpha-\beta)r+\ell(\beta-\gamma)}E^{r+1}_{\alpha,\alpha\gamma+(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha})]\ \ (b=a\,|k|^{\sigma};a=\frac{i\hbar}{2m}) \tag{32}$$

Finally, the required solution (28) is obtained by taking the inverse Fourier transform of the equation (32).

If we set $f(x)=\delta(x)$, where $\delta(x)$ is the Dirac delta function, Theorem 3.1 reduces to the following:

Corollary 3.1 Consider the linear one-dimensional fractional generalization of the Schrödinger equation of a free particle of mass m, defined by

$$_0D_t^{\alpha}N(x,t)+\eta\,_0D_t^{\beta}N(x,t)+\xi\,_0D_t^{\gamma}N(x,t)=\left(\frac{i\hbar}{2m}\right)_{-\infty}D_x^{\eta}N(x,t);$$
$$(0<\alpha\le 1,0<\beta\le 1,0<\gamma\le 1) \tag{33}$$

with initial conditions

$$N(x,0)=\delta(x),\ ,\quad x\in\Re\ ,\quad \lim_{x\to\pm\infty}N(x,t)=0,t>0;\eta,\xi\in R^{+}\ , \tag{33}$$

where $,_0D_t^{\alpha},_0D_t^{\beta},_0D_t^{\gamma}$ are the Caputo fractional derivatives of order $\alpha>0,\beta>0,\gamma>0$, respectively and defined by (15), $_{-\infty}D_x^{\sigma}$ is the Liouville partial fractional derivative of order $\sigma>0$, defined by (21), N(x, t) is the wave function,

$$h = 2\pi\hbar = 6.625 \text{ x } 10^{-27}\ erg\ sec$$
$$= 4.14 \text{ x } 10^{-21}\ MeV\ sec$$

is the Planck constant and f(x) is a prescribed function. Under the above conditions, there holds the following formula for the fundamental solution of (33):

$$N(x,t)=\frac{1}{2\pi}\sum_{r=0}^{\infty}(-1)^r\int_{-\infty}^{\infty}e^{-ikx}\sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell}\{t^{(\alpha-\beta)r+(\beta-\gamma)\ell}E^{r+1}_{\alpha,(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha})$$

$$+\eta t^{(\alpha-\beta)(r+1)+\ell(\beta-\gamma)}E^{r+1}_{\alpha,(\beta)(r+1)+(\beta-\gamma)\ell+1}(-bt^{\alpha})$$

$$+\xi t^{\alpha-\gamma+(\alpha-\beta)r+\ell(\beta-\gamma)}E^{r+1}_{\alpha,\alpha\gamma+(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha})\}dk \quad (b=a\,|\,k\,|^{\sigma};a=\frac{i\hbar}{2m}) \tag{34}$$

provided that the series and integrals in (34) are convergent.

If we set $\alpha=\beta=\gamma=\sigma=\frac{1}{2},$ then Theorem 3.1 yields the following

Corollary 3.2 Consider the linear one-dimensional fractional generalization of the Schrödinger equation of a free particle of mass m, defined by

$${}_0D_t^{1/2}N(x,t)+\eta\,{}_0D_t^{1/2}N(x,t)+\xi\,{}_0D_t^{1/2}N(x,t)=\left(\frac{i\hbar}{2m}\right){}_{-\infty}D_x^{1/2}N(x,t); \tag{35}$$

with initial conditions

$$\mathrm{N(x,\,0)=f(x)},\quad x\in\Re\,,\quad \lim_{x\to\pm\infty}N(x,t)=0,t>0;\eta,\xi\in R^{+}, \tag{36}$$

where ${}_0D_t^{1/2}$ the Caputo fractional derivatives of order $\frac{1}{2}$, defined by (15), ${}_{-\infty}D_x^{1/2}$ is the Liouville partial fractional derivative of order $\frac{1}{2}$ defined by (21), N(x, t) is the wave function,

$$h=2\pi\hbar=6.625\text{ x }10^{-27}\ ergs$$
$$=4.14\text{ x }10^{-21}\ MeVs$$

is the Plank constant and f(x) is the prescribed function, then for the solution of (35), under the above constraints ,there holds the following result:

$$N(x,t)=\frac{(1+\xi+\eta)}{2\pi}\sum_{r=0}^{\infty}(-1)^r\int_{-\infty}^{\infty}e^{-ikx}\sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell}f^{*}(k)E^{r+1}_{1/2,1}(-bt^{\alpha})dk, \tag{37}$$

where $b=a\,|\,k\,|^{\sigma};a=\frac{ih}{2m})$; $h=2\pi\hbar=6.625\text{ x }10^{-27}\ ergs$
$=4.14\text{ x }10^{-21}\ MeVs$

provided that the series and integrals in (37) are convergent.

The following result due to Chaurasia et al. [5] is obtained from the above theorem for $\xi=0$:

Corollary 3.3 Consider the linear one-dimensional fractional generalization of the Schrödinger equation of a free partcle of mass m, defined by

$${}_0D_t^{\alpha}N(x,t)+\eta\,{}_0D_t^{\beta}N(x,t)=\left(\frac{i\hbar}{2m}\right){}_{-\infty}D_x^{\eta}N(x,t);$$
$$(0<\alpha\le 1,0<\beta\le 1,0<\gamma\le 1) \tag{38}$$

with initial conditions

$$\mathrm{N(x,\,0)=f(x)},\quad x\in\Re\,,\quad \lim_{x\to\pm\infty}N(x,t)=0,t>0;\eta,\xi\in R^{+}, \tag{39}$$

where ${}_0D_t^{\alpha}, and {}_0D_t^{\beta}$, are the Caputo fractional derivatives of order $\alpha > 0, \beta > 0$ respectively and defined by (15), ${}_{-\infty}D_x^{\sigma}$ is the Liouville partial fractional derivative of order $\sigma > 0$, defined by (21), N(x ,t) is the wave function,

$$h = 2\pi\hbar = 6.625 \text{ x } 10^{-27} \text{ ergs}$$

$$= 4.14 \text{ x } 10^{-21} \text{ MeVs,}$$

is the Planck constant and f(x) is a prescribed function, then under the above conditions, there holds the following formula for the solution of (38):

$$N(x,t) = \frac{1}{2\pi}\sum_{r=0}^{\infty}(-1)^r \int_{-\infty}^{\infty} e^{-ikx} \sum_{\ell=0}^{r}\binom{r}{\ell}\xi^{\ell}\eta^{r-\ell} f^*(k)\{ t^{(\alpha-\beta)r+(\beta-\gamma)\ell} E^{r+1}_{\alpha,(\alpha-\beta)r+(\beta-\gamma)\ell+1}(-bt^{\alpha})$$

$$+ \eta t^{(\alpha-\beta)(r+1)+\ell(\beta-\gamma)} E^{r+1}_{\alpha,(\alpha-\beta)(r+1)+(\beta-\gamma)\ell+1}(-bt^{\alpha})\} dk \quad , \left(b = a\,|k|^{\sigma}; a = \frac{i\hbar}{2m}\right) \tag{40}$$

provided that the series and integrals in (40) are convergent.

In order to present the results of the next Corollary, we need the following:

Lemma 3.1 If $\Re(\alpha) > 0, \Re(u) > 0, a^2 > 4b,$ then there holds the formula

$$S^{-1}\left[\frac{u^{-2\alpha} + au^{-\alpha}}{u^{-2\alpha} + au^{-\alpha} + b}; t\right] = \frac{1}{\sqrt{(a^2 - 4b)}}\left[(\lambda+\alpha)E_{\alpha}(\lambda t^{\alpha}) - (\mu+\alpha)E_{\alpha}(\mu t^{\alpha})\right], \tag{41}$$

where λ and μ are the real and distinct roots of the quadratic equation $x^2 + ax + b = 0.$

The formula (41) can be established by following the technique developed by Saxena et al. [33]. We have

$$\frac{u^{-2\alpha} + au^{-\alpha}}{u^{-2\alpha} + au^{-\alpha} + b} = \frac{1}{\lambda - \mu}\left[\frac{(\lambda+a)u^{-\alpha}}{u^{-\alpha} - \lambda} - \frac{(\mu+a)u^{-\alpha}}{u^{-\alpha} - \mu}\right], \tag{42}$$

The desired result is obtained by taking the inverse Sumudu transform of both sides of (42).

Now, if we set $f(x) = \delta(x)$, $\xi = 0, \sigma = 2$, α is replaced by 2α and β by α in (28) and use the result (41), we obtain the following result:

Corollary 3.4 Consider the linear one-dimensional linear fractional generalization of the Schrödinger equation

$$\frac{\partial^{2\alpha} N(x,t)}{\partial t^{2\alpha}} + \eta\frac{\partial^{\alpha} N(x,t)}{\partial t^{\alpha}} = b\frac{\partial^2 N(x,t)}{\partial x^2}, 0 < \alpha \le 2 \quad \left(b = ak^2; a = \frac{i\hbar}{2m}\right) \tag{43}$$

with the initial conditions

$$N(x,0) = \delta(x), \quad x \in \Re, N_t(x,0) = 0, \lim_{x\to\pm\infty} N(x,t) = 0, t > 0, \tag{44}$$

where $b \in R, b \neq 0$, $\delta(x)$ is a Dirac-delta function, where $\frac{\partial^{\alpha}}{\partial^{\alpha}}$ and $\frac{\partial^{2\alpha}}{\partial t 2^{\alpha}}$ are the Caputo fractional derivatives of order α and 2α respectively, defined by (15), N(x,t) is the wave function

$$h = 2\pi\hbar = 6.625 \text{ x } 10^{-27} \text{ ergs}$$
$$= 4.14 \text{ x } 10^{-21} \text{ MeVs}$$

Then, for the fundamental solution of (43) under the above constraints, there holds the following formula:

$$N(x,t) = \frac{1}{2\pi\sqrt{(\eta^2 - 4b)}} \int_{-\infty}^{+\infty} \exp(-ikx) \left\{ (\lambda + \eta) E_\alpha(\lambda t^\alpha) - (\lambda + \eta) E_\alpha(\mu t^\alpha) \right\} dk \tag{45}$$

where λ and μ are the real and distinct roots of the quadratic equation

$$y^2 + \eta y + b = 0\,, \tag{46}$$

given by

$$\lambda = \frac{1}{2}(-\eta + \sqrt{(\eta^2 - 4b)}) \text{ and } \mu = \frac{1}{2}(-\eta - \sqrt{(\eta^2 - 4b)}\,, \tag{47}$$

where $b = ak^2, (a = \frac{i\hbar}{2m})$, $E_\alpha(x)$ is the Mittag - Leffler function defined by (13) and provided that the integral in (47) is convergent.

Remark 4.1 A result similar to Corollary 3.4 has been given by Orsingher and Beghin [27], for the fractional telegraph equation.

It is interesting to observe that the Fourier transform of the solution (45) of the equation (44) can be expressed in the form

$$N^*(x,t) = \frac{1}{2} \left\{ (1 + \frac{\eta}{\sqrt{(\eta^2 - 4bk^2)}}) E_\alpha(\lambda t^\alpha) + (1 - \frac{\eta}{\sqrt{(\eta^2 - 4bk^2)}}) E_\alpha(\mu t^\alpha) \right\}, (b = ak^2)\,, \tag{48}$$

where λ and μ are defined in (47) and $E_\alpha(x)$ is the Mittag -Leffler function defined in (13).

If we set $\eta = \xi = 0$, then the Theorem 3.1 gives rise to the following

Corollary 3.5 Consider the linear one-dimensional fractional generalization of the Schrödinger equation of a free particle of mass m, defined by

$${}_0D_t^\alpha N(x,t) = \left(\frac{i\hbar}{2m}\right) {}_{-\infty}D_x^\eta N(x,t), \quad 0 < \alpha \le 1\,, \tag{49}$$

with initial conditions

$$\text{N(x,0)} = \text{f(x)}, \quad x \in \Re\,, \quad \lim_{x \to \pm\infty} N(x,t) = 0, t > 0;\,, \tag{50}$$

where ${}_0D_t^\alpha$ is the Caputo fractional derivatives of order $\alpha > 0$, defined by (15), ${}_{-\infty}D_x^\sigma$ is the Liouville partial fractional derivative of order $\sigma > 0$, defined by (21),and N(x,t) is the wave function,

$$h = 2\pi\hbar = 6.625 \text{ x } 10^{-27} \text{ ergs}$$
$$= 4.14 \text{ x } 10^{-21} \text{ MeVs}$$

is the Planck constant and f(x) is a prescribed function. Then under the above conditions, there holds the following formula for the solution of (48):

$$N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\alpha,1}(-a\,|\,k\,|^{\sigma}t^{\alpha})e^{-ikx}dk,$$
$$= \int_{-\infty}^{\infty} G(x-\zeta,t)f^*(k)d\zeta\,, \tag{51}$$

where the Green function G(x,t) is given by

$$\mathrm{G(x,t)} = \frac{1}{2\pi}\int e^{-ikx}E_{\alpha,1}(-a\,|\,k\,|^{\sigma}\,t^{\alpha})dk$$
$$= \frac{1}{\sigma\,|\,x\,|}H_{3,3}^{2,1}\left[\frac{at^{\alpha}}{(at^{\alpha})^{1/\sigma}}\middle|\begin{matrix}(1,\frac{1}{\sigma}) & (1,\frac{\alpha}{\sigma}) & (1,\frac{1}{2})\\ (1,1) & (1,\frac{1}{\sigma}) & (1,\frac{1}{2})\end{matrix}\right], \tag{52}$$

by virtue of a result given by Haubold et al. [14, p.686, eq.(25)]for evaluating the above integral; where $a = \frac{i\hbar}{2m}$ and $H_{3,3}^{2,1}(.)$ is the H-function defined in the equation (23).

Finally, if we further set $f(x) = \delta(x)$, we obtain another result given by Saxena et al. [35].

4. Conclusion

The method of joint Sumudu transform and Fourier transform is used to solve a fractional generalization of the Schrodinger equation. The solution is expressed in terms of Mittag-Leffler function and the H-function. Several known results follow as special cases of the main result established here. Although, the Sumudu transform is close to the classical Laplace transform, it may be considered theoretical dual to it. Having scale and unit preserving properties, the Sumudu transform may be used to solve problems without resorting to a new frequency domain.

5. References

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