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Representación integral de las $e^{\alpha x}x^k y^l$ – funciones de onda

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Dedicated to Prof. Shyam Kalla
On the occasion of his 76th Anniversary

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Resumen

El trabajo trata con una nueva generalización de funciones analíticas. Se deducen algunas representaciones integrales de las $e^{\alpha x}x^k y^l$ - funciones de onda ($k,l,\alpha - const, > 0$) y sus fórmulas de inversión. Como una aplicación de la teoría se formulan dos problemas y se resuelven estos en forma cerrada.

Palabras clave: Funciones analíticas, funciones de onda.

Integral representation of $e^{\alpha x}x^k y^l$ – wave functions

Abstract

The paper deals with a new generalization of analytical functions. Some integral representations of $e^{\alpha x}x^k y^l$ - wave functions ($k,l,\alpha - const, > 0$), their inversion formulas are derived. As an application of the theory two problems are formulated and solved in the closed form.

Key words: Analytical functions, wave functions.

Introduction

The generalized analytical functions of complex variables appear as a natural and rational generalization of analytical functions.

Picard [1], Beltrami [2], Vekya [3,4], Polozi [5], Manjavidze [6], and others have obtained many important results in the generalization of the theory of analytical functions of elliptic type and their applications. For example, Polozij [5] introduced the analytical functions, using the system:

$$\begin{cases} pu_x - qu_y - \upsilon_y = 0, \\ qu_x + pu_y + \upsilon_x = 0, \end{cases} \tag{1}$$

where p and q are the given real functions of x and y.

Later on the p – analytical and (p,q) – analytical functions found number of applications in different branches of the mathematics, mechanics etc… (axial symmetric theory of elasticity, in the theory of the filtration, solution of the boundary value problems of the theory of rotating covers).

In this paper we study the *p-wave functions* $f(z)=u+i\upsilon$ as the solutions of the following system of the *hyperbolic type*:

$$\begin{cases} pu_x = \upsilon_y, \\ pu_y = \upsilon_x, \end{cases} \tag{2}$$

where $p=e^{\alpha x}x^k y^l$, $(k,l,\alpha$ are positive constants). Some integral representations of p-wave functions and their inversion formulas are constructed.

The p-wave functions describe the processes of mechanics, hydromechanics, the supersonic stream of gas, are useful for solving of the boundary value problems of the mathematical physics etc… Let us notice, that the p-wave functions with the characteristic $p=x^k y^l$ are connected with Euler – Poisson – Darboux equation with two degenerate lines.

Integral representations of the $e^{\alpha x}x^k y^l$ -wave functions

Let G be the domain in the first quarter of the plane $z = x + iy$, bounded by the segments l_1 and l_2 of the real and imaginary axis, respectively, and some curves which are monotone with respect to x and y. The rectilinear segments which are drowned from arbitrary point of the domain orthogonally to axis x and y, belong to the domain G.

Now we state and prove the following theorems related to integral representations of the p-wave functions.

Theorem 1. If $\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)$ is the $e^{\alpha x}y^k$ -wave function [7] in the domain G and

$$\tilde{\varphi}_2(x,y)\big|_{l_2}=0, \tag{3}$$

then the function

$$\tilde{f}(z)=\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)=\int_0^x\left[\tilde{\varphi}_1(\xi,y)x^{1-l}+i\tilde{\varphi}_2(\xi,y)\xi\right]\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi \tag{4}$$

will be $e^{\alpha x}y^k x^l$ -wave function (k,l,α are positive constants) in G, continuous to segment l_2 and satisfies the condition

$$\tilde{\upsilon}(x,y)\big|_{l_2}=0. \tag{5}$$

Proof. According to the conditions of the theorem the functions $\tilde{\varphi}_1(x,y)$ and $\tilde{\varphi}_2(x,y)$ satisfy to next system

$$\begin{aligned} e^{\alpha x}y^k\frac{\partial\tilde{\varphi}_1}{\partial x}&=\frac{\partial\tilde{\varphi}_2}{\partial y}, \\ e^{\alpha x}y^k\frac{\partial\tilde{\varphi}_1}{\partial y}&=\frac{\partial\tilde{\varphi}_2}{\partial x}. \end{aligned} \tag{6}$$

Let us show that for the function (4) the following relations are valid:

$$e^{\alpha x} y^k x^l \frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{\upsilon}}{\partial y},$$
$$e^{\alpha x} y^k x^l \frac{\partial \tilde{u}}{\partial y} = \frac{\partial \tilde{\upsilon}}{\partial x}. \tag{7}$$

In reality:

$$\tilde{\upsilon}_\gamma = \int_0^1 \tilde{\varphi}_{1y}(xt, y)(1-t^2)^{\frac{l}{2}-1} dt,$$

$$\tilde{\upsilon}_x = \frac{\partial}{\partial x}\int_0^1 \tilde{\varphi}_2(xt, y) x^l t (1-t^2)^{\frac{l}{2}-1} dt.$$

Then we have:

$$e^{\alpha x} y^k x^l \tilde{u}_y - \tilde{\upsilon}_x = \int_0^1 \left[e^{\alpha x} y^k x^l \tilde{\varphi}_{1y}(xt, y) - t^2 x^l \tilde{\varphi}_{2xt}(xt, y) \right] \times$$
$$\times \left(1-t^2\right)^{\frac{l}{2}-1} dt - l \int_0^1 x^{l-1} \tilde{\varphi}_2(xt, y) t (1-t^2)^{\frac{l}{2}-1} dt.$$

Taking into account condition $\tilde{\varphi}_2(x, y)\big|_{x=0} = 0$, we get:

$$e^{\alpha x} y^k x^l \tilde{u}_y - \tilde{\upsilon}_x = x^l \int_0^1 \left[e^{\alpha x} y^k \tilde{\varphi}_{1y}(xt, y) - \tilde{\varphi}_{2xt}(xt, y) \right] \left(1-t^2\right)^{\frac{l}{2}-1} dt. \tag{8}$$

Keeping in mind (6) we proved the second relation from (7). The validity of the first relation from (7) is proving analogously.

The validity of the condition (5) follows from the relation:

$$\tilde{\upsilon}(x, y) = \int_0^x \xi \tilde{\varphi}_2(\xi, y)\left(x^2 - \xi^2\right)^{\frac{l}{2}-1} d\xi = \int_0^1 x^l \tilde{\varphi}_2(xt, y) t \left(1-t^2\right)^{\frac{l}{2}-1} dt. \tag{9}$$

Theorem 2. If $\tilde{f}(z) = \tilde{u}(x, y) + i\tilde{\upsilon}(x, y)$ is the $e^{\alpha x} y^k x^l$ - wave function with (5) in the domain G, then the function

$\tilde{\varphi}(z) = \tilde{\varphi}_1(x, y) + i\tilde{\varphi}_2(x, y)$ will be the $e^{\alpha x} y^k$ - wave function and have the following form:
$\tilde{\varphi}(z) = \tilde{\varphi}_1(x, y) + i\tilde{\varphi}_2(x, y)$

$$\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)=\begin{cases}\dfrac{2}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\left\{\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\left[\xi^{l-1}\tilde{u}(\xi,y)\right]}{\left(d\xi^2\right)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}}+\right.\\ \left.+i\dfrac{1}{x}\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\tilde{\upsilon}(\xi,y)}{(d\xi^2)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}}\right\}, \quad l\neq 2n, m=\left[\dfrac{l}{2}\right],\\ \dfrac{2}{(n-1)!}\left\{x\dfrac{d^n}{(dx^2)^n}\left[x^{l-1}\tilde{u}(x,y)\right]+i\dfrac{d^n\tilde{\upsilon}(x,y)}{(dx^2)^n}\right\}, \quad l=2n,\end{cases} \tag{10}$$

and the condition (3) is valid.

Proof. According to (4) $\tilde{\varphi}_1(x,y)$ and $\tilde{\varphi}_2(x,y)$ are the solutions of the following equations, respectively:

$$\tilde{u}(x,y)=x^{1-l}\int_0^x\tilde{\varphi}_1(\xi,y)\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi, \tag{11}$$

$$\tilde{\upsilon}(x,y)=\int_0^x\tilde{\varphi}_2(\xi,y)\xi\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi. \tag{12}$$

The equations (11), (12) are integral equations Abel' type. The solutions of these equations give (10).

Let us show that the function $\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)$ are the $e^{\alpha x}y^k$ - wave function.

Let us introduce the next designations:

$$\begin{aligned}h(x,y)&=e^{\alpha x}y^k\tilde{\varphi}_{1x}-\tilde{\varphi}_{2y},\\ M(x,y)&=e^{\alpha x}y^k\tilde{\varphi}_{1y}-\tilde{\varphi}_{2x},\\ \tilde{L}(x,y)&=e^{\alpha x}y^k x^l\tilde{u}_x-\tilde{\upsilon}_y\\ \tilde{M}(x,y)&=e^{\alpha x}y^k x^l\tilde{u}_y-\tilde{\upsilon}_x.\end{aligned} \tag{13}$$

Then we can rewrite (1) in the following form:

$$\tilde{M}(x,y)=x\int_0^x\tilde{M}(\xi,y)\xi\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi, \tag{14}$$

$$\tilde{L}(x,y)=\int_0^x h(\xi,y)\xi\left(x^2-\xi^2\right).\ d\xi. \tag{15}$$

The solutions of (14) and (15) have the following kind:

$$\tilde{M}(x,y)=\begin{cases}\dfrac{2}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\left[\xi^{-1}M(\xi,y)\right]}{\left(d\xi^2\right)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}},\ l\neq 2n, m=\left[\frac{l}{2}\right],\\ \dfrac{2x}{(n-1)!}\dfrac{d^n\left[x^{l-1}\tilde{M}(x,y)\right]}{(dx^2)^n},\quad l=2n;\end{cases}\tag{16}$$

$$h(x,y)=\begin{cases}\dfrac{2}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\dfrac{1}{x}\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\left[\tilde{L}(\xi,y)\right]}{\left(d\xi^2\right)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}},\ l\neq 2n, m=\left[\frac{l}{2}\right],\\ \dfrac{2x}{(n-1)!}\dfrac{d^n\tilde{L}(x,y)}{(dx^2)^n},\quad l=2n.\end{cases}\tag{17}$$

Because $\tilde{L}(x,y)=0,\ \tilde{M}(x,y)=0,$ then, respectively, $L(x,y)=0,\ M(x,y)=0,$ consequently, $\tilde{\varphi}(z)$ is the $e^{\alpha x}y^k -$ wave function. Let us show, that

$$\tilde{\varphi}_2(x,y)\big|_{l_2}=0\,.$$

i) Let $l=2n$. Using (10) we have:

$$\frac{d^n}{(dx^2)^{n-1}}\left[x^{l-1}\tilde{u}(x,y)\right]=(n-1)!\int_0^x\tilde{\varphi}_1(\xi,y)d\xi,$$

$$\tilde{\varphi}_2(x,y)=\frac{1}{(n-1)!}\frac{d^{n-1}}{(dx^2)^{n-1}}\left[\frac{1}{x}\frac{\partial\tilde{\upsilon}(x,y)}{\partial x}\right].$$

Since $\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)$ is $e^{\alpha x}y^kx^l -$ wave function, then we can rewrite the last relation in the form:

$$\tilde{\varphi}_2(x,y)=\frac{1}{(n-1)!}\frac{d^{n-1}}{(dx^2)^{n-1}}\left[e^{\alpha x}y^kx^{l-1}\frac{\partial\tilde{u}}{\partial y}\right]=$$

$$=\frac{e^{\alpha x}y^k}{(n-1)!}\frac{\partial}{\partial y}\frac{d^{n-1}}{(dx^2)^{n-1}}\left[x^{l-1}\tilde{u}\right]$$

or:

$$\tilde{\varphi}_2(x,y)=e^{\alpha x}y^k\int_0^x\frac{\partial\tilde{\varphi}_1(\xi,y)}{\partial y}d\xi,$$

from here

$$\tilde{\varphi}_2(x,y)\big|_{x=0}=0\,\cdot$$

ii) Let $l \neq 2n$. From (11) we get:

$$\frac{d^{m-1}}{(dx^2)^{m-1}}\left[x^{l-1}\tilde{u}(x,y)\right] = \int_0^x \tilde{\varphi}_1(\xi,y)\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)\left(x^2-\xi^2\right)^{\frac{l}{2}-m} d\xi \cdot$$

According to (10) we have:

$$\tilde{\varphi}_2(x,y) = \frac{2}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\frac{1}{x}\frac{d}{dx}\int_0^x \frac{d^m\tilde{\upsilon}(\xi,y)}{\left(d\xi^2\right)^m}\frac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}},\ l \neq 2n, m = \left[\frac{l}{2}\right].$$

Let us consider

$$\frac{d^m\tilde{\upsilon}(\xi,y)}{\left(d\xi^2\right)^m} = \frac{d^{m-1}}{(d\xi^2)^{m-1}}\left[\frac{1}{2\xi}\frac{\partial\tilde{\upsilon}}{\partial\xi}\right] = \frac{d^{m-1}}{(d\xi^2)^{m-1}}\left[\frac{1}{2\xi}e^{\alpha\xi}y^k\xi^l\tilde{u}_y\right] =$$

$$= \frac{1}{2}y^k e^{\alpha\xi}\frac{\partial}{\partial y}\left[\frac{d^{m-1}(\xi^{l-1}\tilde{u})}{\left(d\xi^2\right)^{m-1}}\right] = \frac{1}{2}y^k e^{\alpha\xi}\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)\times$$

$$\times\int_0^\xi \frac{\partial\tilde{\varphi}_1(\tau,y)}{\partial y}\left(\xi^2-\tau^2\right)^{\frac{l}{2}-m} d\tau.$$

Now we transform $\tilde{\varphi}_2(x,y)$:

$$\tilde{\varphi}_2(x,y) = \frac{y^k\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\frac{1}{x}\frac{d}{dx}\int_0^x \frac{e^{\alpha\xi}\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}}\int_0^\xi \frac{\partial\tilde{\varphi}_1(\tau,y)}{\partial y}\left(\xi^2-\tau^2\right)^{\frac{l}{2}-m} d\tau.$$

Letting $\tau = \xi t, \xi = x\eta$ we get:

$$\tilde{\varphi}_2(x,y) = \frac{\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}y^k\left(3x\int_0^1 e^{\alpha x\eta}\eta^{l-2m+2}\left(1-\eta^2\right)^{m-\frac{l}{2}} d\eta\right)\times$$

$$\times\int_0^1 \frac{\partial\tilde{\varphi}_1(x\eta t,y)}{\partial y}\left(1-t^2\right)^{\frac{l}{2}-m} dt + x^2\int_0^1 \alpha e^{\alpha x\eta}\eta^{l-2m+3}\left(1-\eta^2\right)^{m-\frac{l}{2}} d\eta\int_0^1 \frac{\partial\tilde{\varphi}_1(x\eta t,y)}{\partial y}\left(1-t^2\right)^{\frac{l}{2}-m} dt\Bigg) +$$

$$+\frac{\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}x^2\left(\int_0^1 e^{x\alpha\eta}\eta^{l-2m+3}\left(1-\eta^2\right)^{m-\frac{l}{2}} d\eta\int_0^1\left(-k\frac{1}{y}\frac{\partial\tilde{\varphi}_2(x\eta t,y)}{\partial y}+\right.\right.$$

$$+\frac{\partial^2\tilde{\varphi}_2(x\eta t,y)}{\partial y^2}\Bigg)e^{-\alpha x\eta t}t\left(1-t^2\right)^{\frac{l}{2}-m}dt\Bigg).$$

Hence, $\tilde{\varphi}_2(x,y)\big|_{x=0}=0$. The proof of the theorem is complete.

Definition. The function $u_0(x,y)$ will be called the real wave function in the domain G, if $u_0\in C^2(G)$ and holds the equation:

$$\frac{\partial^2 u_0}{\partial x^2}-\frac{\partial^2 u_0}{\partial y^2}=0 \tag{18}$$

Let us remark the following connection between $e^{\alpha x}y^k-$ wave functions and the wave function [3].

If $u_0(x,y)$ is an arbitrary real wave function in G with the condition

$$\left.\frac{\partial u_0}{\partial y^2}\right|_{y=0}=0 \tag{19}$$

then the integral representation of $e^{\alpha x}y^k-$ wave functions has the next form:

$$\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)=y^{1-k}e^{-\frac{\alpha x}{2}}\int_0^y u_0(x,\tau)\left(y^2-\tau^2\right)^{\frac{k}{2}-1}{}_0F_1\left(\frac{k}{2};-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)d\tau+$$

$$+\frac{i}{k}e^{\frac{\alpha x}{2}}\int_0^y\left[\frac{\partial u_0(x,\tau)}{\partial x}-\frac{\alpha}{2}u_0(x,\tau)\right]{}_0F_1\left(\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{\frac{k}{2}}d\tau, \tag{20}$$

where ${}_0F_1(\nu;z)$ is the partial case of the generalized hypergeometric function [8]

$${}_0F_1(\nu;z)=\Gamma(\nu)\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+\nu)n!}.$$

The solution of the equation (20) with respect to $u_0(x,y)$ has the following form [9]:

$$u_0(x,y)+i\left[\frac{\partial u_0(x,y)}{\partial x}-\frac{\alpha}{2}u_0(x,y)\right]=\frac{2e^{\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)}\times \tag{21}$$

$$\times\frac{d}{dy}\int_0^y\frac{d^m\left[\tau^{k-1}\tilde{\varphi}_1(x,\tau)\right]}{(d\tau^2)^m}{}_0F_1\left(m-\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{m-\frac{k}{2}}\tau d\tau+$$

$$+i\cdot 2\frac{e^{-\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)}\frac{\partial}{\partial y}\left\{\frac{1}{y}\frac{\partial}{\partial y}\int_0^y\frac{d^m\tilde{\varphi}_2(x,\tau)}{(d\tau^2)^m}{}_0F_1\left(m-\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\times\right.$$

$$\left.\times\frac{\tau d\tau}{\left(y^2-\tau^2\right)^{\frac{k}{2}-m}}\right\},\quad k\neq 2n,\ m=\left[\frac{k}{2}\right];$$

as $k=2n$

$$u_0(x,y)+i\left[\frac{\partial u_0(x,y)}{\partial x}-\frac{\alpha}{2}u_0(x,y)\right]=\frac{2e^{\frac{\alpha}{2}x}}{(n-1)!}y\frac{d^n\left[y^{k-1}\tilde{\varphi}_{1n}(x,y)\right]}{(dy^2)^m}+ \tag{22}$$

$$+e^{\frac{\alpha}{2}x}\frac{\alpha y}{(n-1)!}\int_0^y\frac{d^n\left[\tau^{k-1}\tilde{\varphi}_1(x,\tau)\right]}{(d\tau^2)^n}\mathrm{I}_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)\frac{\tau d\tau}{\sqrt{y^2-\tau^2}}+$$

$$+i\frac{2e^{-\frac{\alpha}{2}x}}{(n-1)!}\frac{\partial}{\partial y}\left\{\frac{d^n\tilde{\varphi}_2(x,y)}{(dy^2)^n}+\frac{\alpha}{2}\int_0^y\frac{d^n\tilde{\varphi}_2(x,\tau)}{(d\tau^2)^n}\frac{\mathrm{I}_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau\right\}.$$

Let us construct integral representations of the $e^{\alpha x}y^k x^l$ – wave functions by means of real wave functions.

After some transformations we can write (4) in the following form:

$$\tilde{f}(z)=\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)=x^{1-l}y^{1-k}\int_0^x e^{-\frac{\alpha\xi}{2}}(x^2-\xi^2)^{\frac{l}{2}-1}d\xi\int_0^y u_0(\xi,\tau)\,{}_0F_1\left(\frac{k}{2};-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\times$$

$$\times\left(y^2-\tau^2\right)^{\frac{k}{2}-1}d\tau+ \tag{23}$$

$$+\frac{i}{k}\int_0^x\xi e^{\frac{\alpha\xi}{2}}\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi\int_0^y\left[\frac{\partial u_0(\xi,\tau)}{\partial\xi}-\frac{\alpha}{2}u_0(\xi,\tau)\right]{}_0F_1\left(\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{\frac{k}{2}}d\tau.$$

In order that find the inversion formula of (23) we use (21), (10). We obtain:

$$u_0(x,y)+i\left[\frac{\partial u_0(x,y)}{\partial x}-\frac{\alpha}{2}u_0(x,y)\right]=\frac{4e^{\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(r-\frac{l}{2}+1\right)}\times$$

$$\times\frac{d}{dy}\int_0^y\tau\left(y^2-\tau^2\right)^{m-\frac{k}{2}}\frac{d^m}{(d\tau^2)^m}\left[\tau^{k-1}\frac{d}{dx}\int_0^x\frac{d^r\left[\xi^{l-1}\tilde{u}(\xi,\tau)\right]}{(d\xi^2)^r}\frac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-r}}\right]\times$$

$$\times{}_0F_1\left(m-\frac{k}{2}+1;\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)d\tau+ \tag{24}$$

$$+i\frac{4e^{-\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(r-\frac{l}{2}+1\right)}\frac{\partial}{\partial y}\left\{\frac{1}{y}\frac{\partial}{\partial y}\int_0^y\frac{d^m}{(d\tau^2)^m}\left[\frac{1}{x}\frac{d}{dx}\int_0^x\frac{d^r\tilde{\upsilon}(\xi,\tau)}{(d\xi^2)^r}\frac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-r}}\right]\times\right.$$

$$\times {}_0F_1\left(m-\frac{k}{2}+1;\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\frac{\tau d\tau}{\left(y^2-\tau^2\right)^{\frac{k}{2}-m}}\Bigg\},$$

$$\left(k\neq 2n, l\neq 2r, m=\left[\frac{k}{2}\right], r=\left[\frac{l}{2}\right]\right);$$

$$u_0(x,y)+i\left[\frac{\partial u_0(x,y)}{\partial x}-\frac{\alpha}{2}u_0(x,y)\right]=\frac{4e^{\frac{\alpha}{2}x}}{(n-1)!(p-1)!}y\frac{d^n}{(dy^2)^n}\left[y^{k-1}x\frac{d^p}{(dx^2)^p}\left(x^{l-1}\tilde{u}(x,y)\right)\right]+$$

$$+\frac{2\alpha xye^{\frac{\alpha}{2}x}}{(n-1)!(p-1)!}\int_0^y\frac{d^n}{(dx^2)^n}\left[\tau^{k-1}\frac{d^p}{(dx^2)^p}\left(x^{l-1}\tilde{u}(x,\tau)\right)\right]\frac{I_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau+ \qquad (25)$$

$$+i\frac{4e^{-\frac{\alpha}{2}x}}{(n-1)!(p-1)!}\frac{\partial}{\partial y}\left\{\frac{d}{(dy^2)^n}\left(\frac{d^p\tilde{\upsilon}(x,y)}{(dx^2)^p}\right)+\frac{\alpha}{2}\int_0^y\frac{d^n}{(dx^2)^n}\left(\frac{d^p\tilde{\upsilon}(x,y)}{(dx^2)^p}\right)\frac{I_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau\right\},$$

$\left(k=2n, l=2p, n, p \text{ are integer}\right)$.

The formulas (23), (24) give possibility to reduce the boundary value problems in the class of the $e^{\alpha x}y^k x^l$ – wave functions to the corresponding boundary value problems for the homogeneous wave equation. Let us consider some problems.

In the domain $D=\{(x,y):0<x<\infty, 0<y<\infty\}$ find the $e^{\alpha x}y^k x^l$ – wave functions $\tilde{f}(z)=\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)$, which satisfy the following conditions:

$$\tilde{u}(x,y)\big|_{y=0}=\varphi(x), \quad 0<x<\infty, \qquad (26)$$

$$\tilde{f}(z)\big|_{x=0}=\Phi(y), \quad 0<y<\infty, \qquad (27)$$

where the functions $\varphi(x),\Phi(y)$ are the given continuously differentiable functions.

The solution of this problem we find using (23). For $u_0(x,y)$ we get next boundary conditions:

$$u_0(x,y)=\varphi_0(x)=\begin{cases}\dfrac{4\Gamma\left(\frac{k+1}{2}\right)e^{\frac{\alpha x}{2}}}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(r-\frac{l}{2}+1\right)}\dfrac{d}{dx}\displaystyle\int_0^x\frac{d^r\left[\xi^{l-1}\varphi(\xi)\right]}{(d\xi^2)^r}\frac{\xi d\xi}{(x^2-\xi^2)^{\frac{l}{2}-r}}, l\neq 2n, r=\left[\frac{l}{2}\right],\\ \dfrac{4\Gamma\left(\frac{k+1}{2}\right)e^{\frac{\alpha x}{2}}}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)(n-1)!}x\dfrac{d^r\left[x^{l-1}\varphi(x)\right]}{(dx^2)^n}, l=2n.\end{cases} \qquad (28)$$

$$\frac{\partial u_0(x,0)}{\partial y} = 0, \qquad 0 < x < \infty; \tag{29}$$

$$u_0(0,y) = \Phi_0(y) = \begin{cases} \dfrac{4\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)} \dfrac{d}{dy}\displaystyle\int_0^y \dfrac{d^m\left[\tau^{k-1}\Phi(\tau)\right]}{(d\tau^2)^m} {}_0F_1\left(m-\frac{k}{2}+1; \frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{m-\frac{k}{2}} \tau d\tau, k \neq 2n, m = \left[\frac{k}{2}\right], \\ \dfrac{4\Gamma\left(\frac{l+1}{2}\right)e^{\frac{\alpha x}{2}}}{(n-1)!\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)} y \dfrac{d^n\left[y^{k-1}\Phi(y)\right]}{(dy^2)^n} + \dfrac{2\alpha\Gamma\left(\frac{l+1}{2}\right)}{(n-1)!\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)} y \displaystyle\int_0^y \dfrac{d^n\left[\tau^{k-1}\Phi(\tau)\right]}{(d\tau^2)^n} \times \\ \times \dfrac{I_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}} \tau d\tau, \qquad k = 2n. \end{cases} \tag{30}$$

Using (28) - (30), d'Alembert formula, and formula (23) we obtain the unknown solution.

In the domain $D = \{(x,y): 0 < x < q, y > 0\}$ find the $e^{\alpha x}y^k x^l -$ wave functions $\tilde{f}(z) = \tilde{u}(x,y) + i\tilde{v}(x,y),$ which satisfy the following conditions:

$$\tilde{u}(x,y)\big|_{y=0} = \varphi(x), \qquad 0 < x < q,$$

$$\tilde{f}(z)\big|_{x=0} = \Phi_1(y), \quad \tilde{f}(z)\big|_{x=q} = \Phi_2(y) \quad 0 < y < \infty, \tag{31}$$

where the functions $\varphi(x), \Phi_1(y), \Phi_2(y)$ are the given continuously differentiable functions.

The solution of this problem we find using (23). For real wave function $u_0(x,y)$ we receive the following boundary conditions:

$$u_0(x,0) = \varphi_0(x)$$

$$\frac{\partial u_0(x,0)}{\partial y} = 0, \qquad 0 < x < q; \tag{32}$$

$$u_0(0,y) = \Phi_1^0(y), \quad u_0(q,y) = \Phi_2^0(y), \quad 0 < y < \infty.$$

Using (30) we get $\Phi_1^0(y)$, $\Phi_2^0(y)$.

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