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Revisión de la teoría de Boehmians

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(Dedicated to Professor Shyam Lal Kalla in occasion of his 76 years)

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Resumen

Este artículo está organizado en dos secciones seguidas de una lista de artículos seleccionados de los autores. Probablemente este es el primer intento de escribir con todo detalle acerca de los operadores de Boehmians y J. Mikusinski junto con las contribuciones de varios matemáticos respecto a Bohemians. La teoría de distribuciones de Schwartz fue desarrollada para dar un soporte y fundamentos matematicos comprensibles en la generalización de las propiedades de la función delta de Dirac. A partir del trabajo de Sobolev y Schwartz, se hicieron intentos para generalizar el concepto de distribuciones. Colombeau construyó una nueva algebra diferenciable de funciones generalizadas conteniendo el espacio de distribución, en el cual el producto puede ser definido. El concepto de Boehmians, la mas reciente generalizacion de la teoria de distribuciones de Schwartz, esta motivada por los operadores regulares introducidos y por Boehme. Boehme no adoptó el nombre de Teoria de Bohemians, sino J. Mikusinski y P. Mikusiński fueron inspirados para desarrolar la Teoria de Boehme, que posiblemente adopto el nombre de Boehmians.

Palabras clave: Calculo operacional, operaciones de Mikusinski, funciones generalizadas, distriubuciones de Schwartz, aproximaciones secuencial y funcional, Boehmian, espacio de Beohmian, Boehmian inteprable, boehmian ajustado, ultra Boehmian

Boehmians revisited

Abstract

This article is organized in two sections followed by a list of selected research articles of the authors. Presumably this is the first attempt to write every major and minor detials about Boehmians and J. Mikusinski operators under one cover together with major contributions of various mathematicians with regard to Boehmians. The theory of Schwartz distributions was developed in order to give a concerte and comprehensible mathematical foundation for generalzing the properties of Dirac $\delta -$ function. Starting from the work of Sobolev and Schwartz, attempts were made to generalize the concept of distributions. Colombeau constructed a new differentiable algebra of generalized functions containing the space of distribution, in which product can be defined. The concept of Boehmians, one of the youngest generalization of Schwartz theory of distributions, is motivated by the regular operators introduced by Boehme. Boehme did not, himself, coined the name, the Boehmian, rather J. Mikusiński and P. Mikusiński were inspired to develop the theory of Boehme, which possibly (the conjecture) coined the name Boehmian.

Key words: Operational calculus, Mikusiński operators, generalized functions, Schwartz distributions, sequential and functional approaches, Boehmian, Boehmian space, integrable Boehmian, tempered Boehmians, ultraBoehmian.

On Operational Calculus and Mikusiński Operators

Theory of operators is an apparent prerequisite to study Boehmians and that brings the concept of operational calculus closer to build a better understanding. Plesner [1], while studying the spectral theory of linear operators, reinforced the foundation of operational calculus, which was later extended by Detkin [2]. In the general theory of linear operators, function-operators play an important role. A rule of correspondence established between a set of functions and a class of operators is : *To every function* $F(\lambda)$ *of a given set of functions there corresponds a unit operator* $F(A)$ *and to the unit function* $F(\lambda)=1$*, there corresponds a unit operator* E *and to the function* $F(A)=\lambda$ *an operator* A. Matter of fact, the question relates to the isomorphism between classes of operators and classes of functions, with a unit operator corresponding to a unit function, and the operator A to the function $F(\lambda)=\lambda$, whereas, to the sum and product of functions, $F_1(\lambda)+F_2(\lambda)$ and $F_1(\lambda)F_2(\lambda)$, there correspond the sum and product of corresponding operators.

The use of Laplace transforms restricted the range of applicability of operational calculus techniques, which initiated Jan Mikusiński choose to revert to the original operational view point that did not depend on the Laplace transform. Having started from

$$fg = \int_0^t f(t-\tau)g(\tau)d\tau$$

like Heavisides, he obtained an operational calculus through a straight forward algebraic path. Mikusiński begun from the algebra of functions, where the convolution played the role of product. Even Mikusiński's operational calculus underwent remarks of containing deficiencies because of outright rejection of the Laplace transform which obstructs the realization of some operational formulae. Raevskii could, later, circumvent the difficulty by replacing Mikusiński's expression by a convenient expression

$$fg = \frac{d}{dt}\int_0^t f(t-\tau)g(\tau)d\tau .$$

The nucleus of Mikusiński's reasoning is the idea of the operators, named after him, the theory of which was established during 1950-52. He has represented the genus of fractional number of the type f/g, where f and g are functions in the limit $0 \le x < \infty$. The division (f/g) is understood as an operation, which is the inverse of convolution. If the convolution of two functions f and h is denoted by $f*h$, then $h = f/g$. Polish and German scholars have extended Mikusiński's perceptions. Mikusiński had considered his operators a primitive on an infinite interval. Passage of time in fifties developed the theory of operators on a finite interval, based on the preceding theory.

The algebraic treatment of Mikusiński's operational calculus widened the scope of applications of the techniques of which, according to him, "if the class of functions for which the Laplace transform exists, then two approaches, one due to Mikusiński and the other the Laplace transform technique, are equivalent. However, in the class of functions defined in a finite interval the Laplace or, for that matter, any other transformation, does not reduce a transcendental problem to an algebraic one". Infact, any transformation can not translate the convolution

$$\int_0^t f(t-\tau)g(\tau)d\tau \quad ,$$

with $f(t)=0$, in the first half of the given interval to the usual product, since this convolution equals zero [cf. Mikusiński [3], Shtokalo [4] for more details].

Unlike Mikusiński's first theory of operational techniques (1950-52), which is dealt with (briefly) in the preceding section, the theory propounded here is algebraic in nature and considered as an alternate approach to the problem of constructing a consistent theory of generalized functions [Mikusiński [5]]. It projects the process by which the concept of number is extended from integers to rational numbers and provides a natural approach to operational calculus as well as to generalized functions. Although it (Mikusiński's theory) is successful with functions defined on the positive real line and has been extended to functions of several variables of such type, yet it is not suitable to deal with functions of unrestricted real variable or with functions on an arbitrary region of a space of n dimensions $(n \geq 2)$.

Mikusiński showed that the set $C[0,\infty)$, the letter C suggests continuous, with addition and multiplication by scalars defined in an obvious way and multiplication of two functions a,b of the set defined by the convolution $a*b$, forms a commutative ring, which is called the convolution ring. By virtue of Titchmarsh's theorem [Sneddon [6], page 68], we observe that in this convolution ring, division is a meaningful operation. Familiarity of this is found in when idea of division of integers is dealt with, where division by extending the concept of number from integers to rational numbers in terms of classes of equivalent ordered pairs of integers is ensured. In the present context, we consider ordered pairs of elements of $C[0,\infty)$ and consider (a,b) and (c,d) to be equivalent, if $a*d = c*b$. The class of all ordered pairs of continuous functions, equivalent to (a,b), is denoted by a/b, which is called a convolution quotient.

In C, the set of all convolution quotient, we can define the operations of addition, multiplication by a scalar, and multiplication and show that embedding of $C[0,\infty)$ in C preserves all these operations. That allows, therefore, to write $(a * f)/ a$ as f for any pair of continuous functions a and f and $(\lambda a)/a$ as λ for any scalar λ. The unit element e in C may be written as a/a and it can be shown that multiplication by e reproduces f, which confirms the identification of unit element in C with the Dirac delta function.

The assumption is that the positive real axis is considered ($t \geq 0$). Construction of the rational numbers from the integers is mandatory to know Mikusiński operators and later, the Boehmians; but those who are familiar with this may omit this part. The technicality involves the establishment of the equivalence relations for the ordered pair of integers, the separation into equivalence classes, and the verification of the independence of choice of representative of an equivalence class, for instance 6/5 and 30/25 are both representative for the class of quotients which are equivalent to 6/5. Now care must be taken to construct the field of quotients, the naming of equivalence classes as rational numbers, and embedding of the integers into this new system. We write a/b (for notations) to represent the elements of the new system, where a and b are integers. The second element is not always zero. The operations of addition and multiplication is simple matter for that cause. We observe that b/b plays the role of unit element in the new system and that $0/b$ plays the role of zero. If $a \neq 0$, then the equation $(a/b)\,(x/y) = (c/d)$ has the solution $(bc)/(ad)$ and it is unique. All these and little more make available the operational calculus in which division is possible. In reference to what is written above, we obtain a model for the construction of the convolution quotient from the continuous functions by the construction of the rational numbers form the integers. Set of functions, which are continuous for $t \geq 0$ are considered, the addition that is taken is pointwise. Since pointwise multiplication does have non-trivial divisors of zero, whereas convolution does not have so, the multiplication considered is convolution. We point out that equivalence class (mentioned above) is named as Mikusiński operators or as generalized functions.

For $g \neq \{0\}$, which is the constant function, the convolution quotient g/g must appear, which leads us to a unit element among the generalized functions. It may be noted that the subset of the Mikusiński operator satisfies all of the properties of real (complex) numbers as well as the operational properties of continuous functions, e.g. addition, multiplication by real (complex) numbers and convolution, which are required for multiplication of functions by numbers. As a consequence, within the field of Mikusiński operators (generalized functions) we can now consider addition of a number and a function, see Buschman [7] for relevant information.

We should have complied with the regulation of describing the generalized functions or the Schwartz theory of distributions, of which the Boehmian is called the youngest generalization, prior talking about Boehmians. We have, at the last moment, negated this idea for the simple fact that generalized functions have wide familiarity among readers, and moreover, due to paucity of space. However, we give a brief note on the generalized functions prior moving to the section(s) only on Boehmians.

There are some problems encountered in applied mathematics when transform methods are applied to analyze physical situations in which impulsive forces or point sources are involved. Introduction of Dirac delta function simplifies formal calculations. But the rules for doing the manipulation do not follow, in natural way, the methods of classical analysis. This, possibly, led to the advent of the concept of generalized functions. Bochner [8] and Sobolev [9] have coined first ideas of such an approach but the firm foundation was put by the work of Schwartz [10], culminating in the publication of his treatise. Zemanian [11, 12] exhibit excellent study on this concept.

Paul Dirac [13] introduced, for the first time, in quantum mechanical studies, the delta function which possesses the property $\delta(x)=0, x\neq 0$ and $\delta(x)\varphi(x)dx=\varphi(0), \varphi\in C$. It was soon pointed out by mathematicians that from purely mathematical point of view this definition is meaningless. It was, of course, even clear to Dirac himself that the δ - function is not a function in the classical sense and, what is important, it operates as an operator (more precisely as a functional) that relates, via above formula to each continuous function φ a number $\varphi(0)$, which is its value at a point O.

The simplest attempt at such a generalization,i.e., to generalize the entire concept of a function, is due to Mikusiński [14] which is developed by Temple [15, 16]. This method defines generalized functions as classes of equivalent fundamental sequences of continuous functions, which is similar to that used when real numbers are introduced with the help of fundamental sequences of rational numbers. References for further reading, among many others, are Beltrami and Wohlers [17], Bremermann [18], Carmichael and Pilipovic [19], Debnath [20], Debnath and Mikusiński [21], Erdélyi [22], Friedmann [23], Gel'fand and Shilov [24, 25, 26, 27, 4 vols.], Hoskins [28], Korevarr [29], Mikusiński and Sirorski [30], Pandey [31], Zemanian [11,12]. Distributions are generalization of locally integrable functions on the real line, or more generally a generalization of functions which are defined on an arbitrary open set in the Euclidean space. The mathematical theory called the theory of distributions, which enabled the introduction of the Dirac delta function without any logical restrictions, was coined in forties of the preceding century. As once the theory of real number was generalized, this theory generalized the notion of function.

The two most important approaches in theory and practice are: functional approach advented by Soboleff [9] and Schwartz [32] where distributions are defined as linear functionals continuous in linear spaces; sequential approach given by Mikusiński [14] , where distributions are defined as class of equivalent sequences. It may be noted that among important, in practice operations, are regular and non-regular operations. For example, the two argument operations of product $A(\varphi,\psi)=\varphi\cdot\psi$ and the convolution $A(\varphi,\psi)=\varphi*\psi$ are not regular operations and, therefore, they cannot be defined for arbitrary distributions. Mikusiński [33, 34] devised a general method to define irregular operations on distributions (see also Antosik et al. [35]).One may also refer to Mikusiński [36], Kaminski [37], Antosik and Ligeza [38] among others.

The alternate approach is to study distributions as limit of sequences of functions. Logical construction of such limits is based on the Cantor's concept of equivalence classes. To each distribution in the functional approach there corresponds one distribution in the sequential approach, and conversely. The approach is to establish, first, the usual property of the distribution as a derivative of continuous function and then develop the remaining on the basis of both, as a limit of continuous functions and as a derivative of a function of a distribution [cf. Lojasiewicz [39] and Zielenzny [40]].

The fundamental term to form the basis of the sequential approach is the identification principle. Oriented segments x and y are said to be equivalent if they are parallel and have the same length and orientation, we write $x : y$, which has the properties, (i) $x \sim x$ (reflexive), (ii) if $x \sim y$, then $y \sim x$ (symmetric) and (iii) if $x \sim y$, $y \sim z$, then $x \sim z$ (transitive). By means of equivalence relation we obtain a decomposition of the set of all oriented segments into disjoint classes such that the segments in the same class are equivalent and in different classes they are not, by virtue of Cantor's definition of real numbers. Rational numbers are the basic concepts for understanding Cantor's theory, the functions continuous in $A < x < B$ $(-\infty \le A < B \le \infty)$ are the starting point for the theory of sequential approach. A sequence $\{F_n(x)\}$ of continuous functions $(A < x < B)$ is called a fundamental sequence if there exists a sequence $\{F_n(x)\}$ and an integer $k \geq 0$ such that $F_n^{(k)} = f_n(x)$ and the sequence $\{F_n(x)\}$ converges almost uniformly.

If there exists sequences $\{F_n(x)\}$ and $\{G_n(x)\}$ and an integer $k \geq 0$ such that

(i) $F_n^{(k)} = f_n(x)$ and $G_n^{(k)} = g_n(x)$

(ii) $F_n(x) \Rightarrow\Leftarrow G_n(x)$,

then the fundamental sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are said to be equivalent, we write $\{f_n(x)\} \sim \{g_n(x)\}$. In other words, fundamental sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are equivalent if and only if the sequence given by $f_1(x), g_1(x), f_2(x), g_2(x), \ldots$ is fundamental and moreover, in that case there exists an integer $k \geq 0$ and the continuous functions $F_n(x)$ and $G_n(x)$ such that $F_n^{(k)} = f_n(x)$, $G_n^{(k)} = g_n(x)$ and the sequence $F_1(x), G_1(x), F_2(x), G_2(x), \ldots$ converges uniformly and, consequently, (i) and (ii) hold true. By virtue of conditions (i) - (iii) (i.e. reflexive, symmetric and transitive as dfined above), the set of all fundamental sequences $\{f_n(x)\}, A < x < B$, is partitioned into equivalence classes without common elements such that two fundamental sequences are in the same equivalence class if and only if they are equivalent, which (the equivalence classes) will be called distribution in $A < x < B$. The notion of the distributions is, thus, obtained, from the identification of equivalent fundamental sequences, those distributions are denoted by $[f_n(x)]$.

Denoting by 0, is the zero distribution, which is the distribution coinciding with the function identically equal to zero, we mean $0 + f(x) = f(x)$ and $0 \cdot f(x) = 0$. The symbol 0 has two interpretations, for the former it means number zero and for the latter, it is zero distribution, Loonker [41] and Krystyna [42]. A formal definition of distribution was due to Mikusiński [43], based on which Mikusiński and Sikorski [30, 44] developed sequential theory of distributions and later Mikusiński and Antosik [35] wrote the monograph. Mikusiński's definition of distributions in sequential sense is analogue to the definition of real numbers in the Cantor's theory.

The operations, addition of smooth functions, difference of smooth functions, multiplication of a smooth function by a fixed number $\lambda;\lambda\varphi$, translation of the argument of a smooth function $\varphi(x+h)$, derivation of a smooth function of a fixed order m; $\varphi^{(m)}$, multiplication of a smooth function by a smooth fixed function ω; $\omega\varphi$, substitution of a fixed smooth function $\omega \neq 0$, product of a smooth function with separated variables; $\varphi_1(x)\varphi_2(y)$ convolution of a smooth function with a fixed function ω from the space D (of smooth functions whose supports are bounded); $(f \bullet \omega)(x) = \int_{\mathrm{R}} \varphi(x-t)\omega(t)dt$, inner product of a smooth function with a fixed function form the space Δ; $(\varphi, \omega) = \int_{\mathrm{R}} \varphi(x)\omega(x)dx$, are all regular. An advantage of the sequential approach to the theory of distributions is the simplified way of extending to distributions many operations which are regular. Moreover, we know that every distribution is locally a distributional derivative of a finite order of a continuous function. It is also a natural consequence that the sequence $(f \bullet \delta_n)$ is distributionally convergent to f (i.e., fundamental for f) for an arbitrary distribution $f \in \mathrm{R}^1$, where (δ_n) is the delta sequence.

Jan Mikusiński also explored the theory of integration. His definition of the Lebesgue integral is simple and possesses clean geometrical meaning, which can be formulated for functions defined in

R^k with values in a Banach space which yields a uniform approach to the Lebesgue and the Bochner integrals (both have a great cohesiveness to Boehmians). The function $f:\mathsf{R}^k \to X$ $(f:\mathsf{R}^k \to \mathsf{R}^1)$, where X is a Banach space, is called Bochner (Lebesgue) integrable if there exists a sequence of interval $I_n = [a_{1n}, b_{1n}) \times \ldots \times a_{kn}, b_{kn})$ in R^k in R^k and a sequence $(\lambda_n)_{n\in N}$ of elements of X such that

$$\sum_{n=1}^{\infty} |\lambda_n| \text{ vol}(I_n) < \infty$$

and

$$f(x) = \sum_{n=1}^{\infty} \lambda_n \chi I_n(x),$$

at those points x at which the series is absolutely convergent, where χ_I denotes the characteristic function of an interval I.

Bochner (Lebesgue) integral of a function satisfying above conditions, is defined by

$$\int f = \sum_{n=1}^{\infty} |\lambda_n| \text{ vol}(I_n),$$

which is equivalent to the classical definitions of Bochner and Lebesgue integrals, see Mikusiński [45].

Introducing Boehmians

That we are writing in this section is one of the youngest generalizations of functions and more particularly that of Schwartz theory of Distributions, devised by Thomas Kalman Boehme, descendant of Prof. Arthur Erdélyi, who earned his degree of Ph.D. from California Institute of Technology in 1960. Instead of writing on Boehmians and the Boehmian space straightway, we desire to mention, very briefly, some relevant and fruitful thoughts given in the Thesis of Boehme [46]. In his thesis, the finite part of divergent convolution integrals is studied and explored by utilizing Mikusiński's operational calculus (possibly that is the coining of the idea for Boehmians). In Chapters 2 and 3, the concept of an analytic operator function is utilized. An operator function $f(z)$ is said to be an analytic operator function on an open region $\mathbf{S}$ of the complex plane if there is an operator $a \neq 0$ such that $af(z) = \{af(z,t)\}$ has a partial derivative with respect to z, which is continuous on $\mathbf{S} \times 0, \infty)$. Let $f(z)$ be an analytic operator function and $\{f(z,t)\}$ is a continuous function on $S \times [0, \infty)$. Suppose also that for each $t > 0$, $f(z,t)$ is an analytic function on z on larger region $\mathbf{S}^* \supset \mathbf{S}$. Let $f^*(z)$ is an analytic operator function on $\mathbf{S}^*$ such that $f^*(z) = f(z)$ on $\mathbf{S}$. Then the operator function $f^*(z)$ is called [FP $f(z,t)\}$ on $\mathbf{S}^*$.

In fact, the use of the finite parts of divergent integrals started with Cauchy who used, what he called "intégrale extraordinaire", to give a sense to the gamma function for negative values of the argument. This notion has been used and extended by various authors, among them are Schwartz [10] and Lighthill [47] who have applied the theory of distributions to extend the idea of the finite part of divergent integrals. Butzer [48] used the Mikusiński's operational calculus to study the finite part of the divergent convolution integrals.

For certain functions $\{f(z,t)\}$, [cf. Boehme [46, Chap.3]], the finite part of the convolution integral $\int_0^t g(t-u) f(z,u)du$ has been defined by Hadamard [49] and Bureau [50] even though for some values of z, the function $\{f(z,t)\}$ is not a Lebesgue integrable function.

The idea of construction of Boehmians is coined from the concept of regular operators introduced by Boehme [51], which form a subalgebra of the field of Mikusiński operators and they, thus, include only such functions whose support is bounded from left. Mikusiński and Mikusiński [52] attempted to generalize the notion of regular operators so as to include all continuous functions and to formulate a general construction of Boehmians. Strictly speaking, the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions. Mikusiński [53] introduced and studied the convergence of Boehmians, where the space furnished with the induced convergence, appears to be a complete quasi-normed space. For every ring without zero divisors, there exists the corresponding field of quotients.

The space C^+ of all continuous functions on the real line R with supports bounded from left forms a ring without zero divisors with respect to the convolution. The field of quotients for the space C^+ is called (usually) the field of Mikusiński operators, which is when replaced by the space of all continuous functions C, the construction of the field of quotients becomes impossible due to the presence of zero divisors in C. The construction of Boehmians is similar to that of the field of quotients and in some cases, it is interesting to note, it gives just the field of quotients. On the other hand, the construction of Boehmian is possible where there are zero divisors, such as the space C.

Let G be a linear space and S be the subspace of G. Let to each pair of elements $f \in G$ and $\varphi \in S$, the product $f * g$ is assigned ($*$ is a map from $G \times S$ to G) such that

(i) if $\varphi, \psi \in S$, then $\varphi * \psi \in S$ and $\varphi * \psi = \psi * \varphi$

(ii) if $f \in G, \varphi, \psi \in S$, then $(f * \varphi) * \psi = f * (\varphi * \psi)$

(iii) if $f, g \in G, \varphi \in S$, and $\lambda \in R$, then

$(f + g) * \varphi = f * \varphi + g * \varphi$

and $\lambda(f * \varphi) = (\lambda f) * \varphi$.

Let Δ be a family of sequences of elements from S such that

(iv) if $f, g \in G, (\delta_n) \in \Delta$ and $f * \delta_n = g * \delta_n$ $(n = 1, 2, \ldots)$, then $f = g$.

(v) if (φ_n) $(\delta_n) \in \Delta$, then $(\varphi_n * \delta_n) \in \Delta$.

Elements of Δ will be called delta sequence. Consider the class A of pair of sequences defined by $\mathsf{A} = \{(f_n), (\varphi_n) : (f_n) \subseteq G^N, (\varphi_n) \in \Delta\}$, for each $n \in N$. An element $((f_n), (\varphi_n)) \in \mathsf{A}$ is called quotient of sequences, denoted by f_n / φ_n, if

$$f_i * \varphi_j = f_j * \varphi_i \, , \ \forall i, j \in N \ .$$

Two quotients of sequences f_m/φ_m and g_n/ψ_n are called equivalent, denoted by $f_m/\varphi_m \sim g_n/\psi_n$, if

$$f_m * \psi_n = g_n * \varphi_m \, , \ \forall m, n \in N \ ,$$

which splits A into equivalence classes, of which the class containing f_n/φ_n is denoted by $[f_n/\varphi_n]$ These equivalence classes are called Boehmians and the space of them is denoted by $B = B(G, \Delta)$ Following illustrates the behaviour of Boehmians for the algebraic properties.

(i) The sum of two Boehmians and the multiplication by a scalar are defined by

$$[f_n/\varphi_n]+[g_n/\psi_n]=[(f_n*\psi_n)+(g_n*\varphi_n)/(\varphi_n*\psi_n)]$$

and

$$[\alpha f_n/\varphi_n]=[\alpha f_n/\varphi_n],\ \alpha\in C.$$

(ii) The operation $*$ and the differentiation are, respectively, defined by

$$[f_n/\varphi_n]*g_n/\psi_n]=[(f_n*g_n)/(\varphi_n*\psi_n)]$$

and

$$D^\alpha[f_n/\varphi_n]=[D^\alpha f_n/\varphi_n].$$

In particular, if $[f_n/\varphi_n]\in B$ and $\delta\in S$ is any fixed element, then the product $*$ is defined by

$$[f_n/\varphi_n]*\delta=[(f_n*\delta)/\varphi_n],$$

which is said to be in $B(G,\Delta)$.

More often G, which is also the quasi-normed space, is found to be equipped with the notion of convergence. The intrinsic relationship between this notion of convergence and the product $*$ are given by

(i) if $f_n\to f$ as $n\to\infty$ in G and $\varphi\in S$ be any fixed element, then $f_n*\varphi_n\to f\ *\varphi$ as $n\to\infty$ in G.

(ii) if $f_n\to f$ as $n\to\infty$ in G and $(\delta_n)\in\Delta$, then $f_n*\delta_n\to f$ as $n\to\infty$ in G.

In the Boehmian space B the δ - and Δ - convergences are stated as:

(i) A sequence of Boehmians (x_n) in the Boehmian space B is said to be δ - convergent to a Boehmian x in B, which is denoted by $x_n\xrightarrow{\delta}x$ if there exists a delta sequence (δ_n) such that $(x_n*\delta_n)$, $(x*\delta_n)\in G, \forall n\in N$ and $(x_n*\delta_k)\to(x*\delta_k)$ as $n\to\infty$ in G, $\forall k\in N$.

(ii) A sequence of Boehmians (x_n) in B is said to be Δ- convergent to a Boehmian x in B, denoted by $x_n\xrightarrow{\Delta}x$ if there exists a delta sequence $(\delta_n)\in\Delta$ such that $(x_n-x)*\delta_n\in G,\forall n\in N$ and $(x_n-x)*\delta_n\to 0$ as $n\to\infty$ in G.

Suppose U is an open set. Then a Boehmian $x\in B$ is said to vanish on U if for each compact set $K\subseteq U$ there exists a representative f_n/φ_n of x such that $f_n=0$ on K for each $n\in N$. Thus, the support of a Boehmian x is defined as the complement of the largest open set on which x vanishes. In what follows is an example of a Boehmian space in which the distributions D' can be imbedded.

Consider $G=C^\infty(\mathbb{R})$, which is equipped with the topology of uniform convergence on compact set $S=D(R)$. Let Δ be the class of sequences from D, which satisfies the conditions $\int_{\mathbb{R}}\delta_n(x)xd=1$, $\int_{\mathbb{R}}|\delta_n(x)|\leq M$ and supp $\delta_n\to 0$ as $n\to\infty$. For $f\in G,\varphi\in S$, the convolution $*$ is defined by $(f*\varphi)=\int_{\mathbb{R}}f(x-t)\varphi(t)dt$. Indeed, $*$ defines a map from $G\times S$ to G and a member of Δ satisfies the conditions

(i) if $\alpha,\beta\in G,(\delta_n)\in\Delta$ and $(\alpha*\delta_n)=(\beta*\delta_n)$ for each $n\in N$, then $\alpha=\beta$ in G, and

(ii) if (δ_n), $(\varphi_n)\in\Delta$, then $(\delta_n*\varphi_n)\in\Delta$,

and thereby generates a Boehmian space, which is $B=B(C^\infty(\mathbb{P}\ ),\Delta)$, members of which are called C^∞-Boehmians. In another case, consider G to be set of all locally integrable functions on R and identify two such functions, whenever they are equal almost everywhere with respect to the usual Lebesgue measure on $\mathbb{R}$, the topology of which is taken to be the semi-norm topology, generated by

$$p_n(f) = \int_{-n}^{n} |f| d\lambda \quad , n = 1,2,\ldots .$$

Also consider $S = \mathrm{D}(\mathrm{R})$ and Δ is the class of sequence from D (discussed in preceding sections). Then a corresponding Boehmian space $B = B(G, \Delta)$ is obtained, called the space of locally (or local) Boehmians. $\mathrm{D}'(\mathrm{R})$ can be imbedded continuously in both the above mentioned Boehmian spaces in the sense that the map $\mathrm{D}' \in B$, given by $u \to (u * \delta_n) / \delta_n]$, defines a one-to-one function in such a way that $u_m \to u$ in D' implies $x_m \to x$ in B, where $x = [(u_m * \delta_n) / \delta_n]$ and $x = [(u * \delta_n) / \delta_n]$.

Mikusiński [54] has constructed a Boehmian space B_{L_1} consisting of integrable Boehmian, on which the Fourier transform is defined as a continuous function. The Boehmian space B_{L_1} is constructed due to $G = \mathrm{L}(\mathrm{R})$ and the class Δ, which satisfies the conditions

$$\int_{\mathbb{R}} \delta_n(x) dx = 1 \ , \ \forall n \in N$$

$\|\delta_n\| < M$, for some $M \in \mathbb{R}$ and all $n \in N$

$\lim_{n\to\infty} \int_{|x|>\varepsilon} |\delta_n(x)| dx = 0$, for each $\varepsilon > 0$

and $(f * \varphi) = \int_{\mathbb{R}^n} f(x-y) g(y) dm(y)$,

where * is the convolution, except for the use of ordinary Lebesgue measure, in place of normalized Lebesgue measure. Mikusiński [54] has also shown that whenever $[f_n/\varphi_n] \in B_{L_1}$,

$$\hat{f}_n(x) = \int_{\mathrm{R}} f_n(t) e^{-itx} dt \quad ,$$

converges uniformly on each compact set in R. Then the Fourier transform of an integrable Boehmian $[f_n/\varphi_n]$ is defined as the limit of $\{\hat{f}_n\}$ in the space of continuous functions on R. Mikusiński [55, 56] suggested an extension of space of the Fourier transformable Boehmian containing the tempered distribution S'. The space of tempered Boehmians, which is denoted by B_T, is constructed by taking $G = T$, which is the space of slowly increasing functions on R. Note also that every distributions is the Fourier transform of a tempered Boehmian.

In what follow is the published research work of authors (of this article) related to Boehmians. Looking into both aspects the paucity of space and degree of tolerance of the reader, only the abstract of each paper is given, without destroying the inquisitive thirst of the reader.

1. <u>On the Mellin transform of tempered Boehmians</u>, U.P.B. Sci. Bull. Series A, 62 (4)(2000), 39-48.

Two theorems have been proved on the characteristic theme, that the Mellin transform of tempered Boehmian is a Schwartz distribution. The Mellin transform $\hat{f}(is)$ of slowly increasing function f is the distribution, given by

$$\left\langle \hat{f}(is), \overline{\hat{\varphi}(is)} \right\rangle = 2\pi \left\langle f(e^x), \overline{\varphi(e^x)} \right\rangle .$$

The Mellin transform $\hat{F}$ of tempered Boehmian $F=[f_n/\varphi_n]$ is the limit of $\{\hat{f}_n\}$ in $\mathcal{D}'$ (the space of distributions). Statement of one of the two theorems proved is, *If* $[f_n/\varphi_n]\in B_I$ *, then the sequence* $\{\hat{f}_n\}$ *converges in* $\mathcal{D}'$*. Moreover, if* $[f_n/\varphi_n]=[g_n/\gamma_n]\in B_I$ *, then the sequences* $\{\hat{f}_n\}$ *and* $\{\hat{g}_n\}$ *have the same limit for the Mellin transform of tempered Boehmians*.

2. Wavelet transform of the tempered Boehmians, Hadronic J. Suppl. 18 (2003), 403-410.

This paper deals with the extension of tempered distribution to a class of Boehmians known as tempered Boehmians and defined it on the wavelet transform. Central theme being proving that the wavelet transform of a tempered Boehmian is a distribution, i.e., we have characterized the distributions of the transformable Boehmians. The inversion theorem is also proved. In proving the theorems, continuous Gabor transform (or windowed Fourier transform) of f is used [cf. Debnath [57]] and then the Parseval formula for the Gabor transform is invoked.

3. Wavelet transform for integrable Boehmians (with Lokenath Debnath), J. Math. Anal. Appl. 296 (2) (2004), 473 - 478 .

By applications of continuous wavelet transform and invoking Burzyk's conjecture, the wavelet transform for integrable Boehmians is obtained. Inversion theorem is also proved. Wavelet transform is [cf. Koorwinder [58]]

$$(\Phi_g f)(a,b)=\int f(x)\overline{g_{a,b}(x)}dx=F(a,b) \quad ,$$

where $f\in L^2(\mathbb{R}^d), a\in\mathbb{R}^*, b\in\mathbb{R}^d, \mathbb{R}$ is a set of real numbers, d=1, $\mathbb{R}^*=\mathbb{R}/\{0\}, (\Phi_g f)=(f*h_{a,0})(b)$, and $h(x)=\overline{g(-x)}$.

4. Ultradistribution and ultra-Boehmian of wavelet transform (with S. L. Kalla), Hadronic Journal, 29 (2006), 485-496.

We have investigated certain testing function space for the wavelet transform. Also obtained are ultradistribution and ultra-Boehmians for the wavelet transform. Section 2 deals with the testing function space Z of the wavelet transform, Section 3, based upon the statement (Theorem proved there) that, *the space of all ultradistributions* Z' *contains the space* S' *of tempered distributions*, establishes the result for ultradistrbution of wavelet transform. While extending the wavelet transform to the ultra-Boehmian space in Section 4, it is proved that, *if* $[f_n/\varphi_n]\in\beta_z$*, then the wavelet transform converges in* $\mathcal{D}'$, and further, *if* $[f_n/\varphi_n]=[g_n/\gamma_n]$ *belongs to* β_z*, then the wavelet transform converges to the same limit to which do ultra-Boehmians*.

5. The Cauchy representation of integrable and tempered Boehmians, Kyungpook Math. J. 47 (2007), 481-493.

The paper proves results based on the concept that, a relation between the Cauchy representation of the Fourier transform of the functions in L_2-space and a decomposition of the Fourier transform into two parts, each of which gives an analytic function in the half plane, define that the decomposed transform is convergent for classes of functions larger than those in L_2-space .Section 2 investigates the Cauchy representation of integrable Boehmians by invoking the relation between the Cauchy representation and the Fourier transform and using properties of the former in L_1-space. In Section 3, we have investigated the Cauchy representation of tempered Boehmians. Inversion formulae, for results in Section 2 and 3, are also proved. The conclusive remark of the paper is, the Cauchy representation of an integrable Boehmian and the tempered Boehmian is a distribution.

6. Hilbert Transform for Lacunary Boehmians, Global J. Math. Anal. 1 (1-2)(2007), 85-90.

A series of the form $\sum_{n=-\infty}^{\infty} a_n \exp(i\lambda_n t)$, where $\{\lambda_n\}$ is a sequence of positive integers for which $\inf(\lambda_{n+1}/\lambda_n) > 1$ and $\lambda_{-n} = -\lambda_n$ for all $n \in N$, is called a lacunary series. Nemzer [59] has investigated the space of the lacunary Boehmians, which has lacunary Fourier series representation. In the present paper we study the Hilbert transform for the lacunary Boehmians. A sequence of positive integers $\{\lambda_n\}_{n=1}^{\infty}$ is called Hadamard-lacuanry or simply lacunary if there exists a constant $q > 1$ such that $\lambda_{n+1} > q\lambda_n$ for all n.

7. Mellin transform of fractional integrals for integrable Boehmians, J. Indian Math. Soc. 74 (1-2) (2007), 83-89.

The Riemann-Liouvelle fractional integrals, [Samko et al. [60]], for a function $\varphi(x) \in L_1(a,b)$ are extended from finite interval $[a,b]$ to half axis [Samko et al. [60], page 94] by the formula

$$(I_{0+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)dt \quad ,0 < x < \infty$$

the Mellin transform of which is [cf. Podlubny [61], page 115]

$$(I_{0+}^{\alpha}\varphi)(x) = \frac{x^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\varphi(x\xi)g(\xi)d\xi \quad .$$

8. Generalized Stieltjes transform and its fractional integrals for integrable Boehmians, Austral. J. Math. Anal. Appl. 5 (1) (2008), 1-8

Using the distributional Stieljes transform and the Parseval relation for the generalized stieltjes transform, in Section 2, a lemma is proved which is further used in proving an important theorem. Section 3 exhibits use of fractional integral operators for integrable Boehmians.Stieltjes transform of fractional integral opeartor is investigated for integrable Boehmians which shows that the Stieltjes transform of fractional integral operator for an integrable Boehmian $F = [f_n/\delta_n]$ is defined as the limit of $(\mathsf{G}_p I_{0+}^{\alpha} f_n)$, which is the space of continuous functions on R.

9. Fourier sine (cosine) transform for ultradistributions and their extensions of tempered and ultra-Boehmian spaces (with S. K. Q. Al-Omari and S. L. Kalla), Integral Transforms Spl. Fuct. 19 (6) (2008), 453-462.

This paper has regarded ultradistributions for the Fourier sine (cosine) transform on certain dual testing space and extension of them on tempered and ultra-Boehmian spaces.

We conclude with a remark that for explanations of notations used and detailed calculations of the results therein, one may refer to original papers, mentioned above. It may not be out of place to mention that some of our work, which are not included above, are viz. Banerji and Loonker [62], Banerji [63], Loonker and Banerji [64, 65, 66, 67] and Banerji and Loonker [68] , Singh et al. [69], Loonker and Banerji [70], Singh and Banerji [71, 72], Loonker and Banerji [73], Singh et al. [74].

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